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Izvestiya Akademii Nauk SSSR,
Seriya Mekhanika i Mashinostroyeniye,
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LOCAL STABILITY OF SANDWICH SHELLS OF REVOLUTION

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A mathematical treatment is given of the linearized stability of sandwich shells of revolution, with asymmetric structure and a rigid core under compound stress. Stability and mechanical strength formulas are derived for the shear exerted by the core on the load-carrying faces, for free and clamped ends of shells of conical, cylindrical, and spherical shape.

Extensive literature is devoted to the stability of sandwich shells; instances of the symmetric structure of shell thickness in shells with a lightweight core have been been investigated in considerable detail.

The discussion of these studies is a separate task in itself. Below are presented, on the basis of the equations derived elsewhere (Bibl.1), the results of a study of the linearized stability of sandwich shells with asymmetric structure and a rigid core (shell of revolution, spherical, cylindrical, and conical shells) under compound stress. Definite results (Bibl.2, 3) are obtained on the basis of a specific case.

1. Fundamental Equations

As shown before (Bibl.1), the equations of the finite buckling of cambered elastic thin sandwich shells, with an asymmetric structure containing a rigid transversally isotropic core enclosed between isotropic load-carrying faces may be written with respect to the force function F and displacement function χ. We reproduce the system of these equations, complemented by terms taking into account the initial irregularities, on the assumption that the tangential surface loads have the potential Ψ

$$\nabla^2\nabla^2 F - (1-\nu)\nabla^2\Psi = \mathrm{Eh}\Big[k_{11}\frac{\partial^2 w}{\partial y^2} + k_{22}\frac{\partial^2 w}{\partial x^2} + \Big(\frac{\partial^2 w}{\partial x\partial y}\Big)^2 + 2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 w_0}{\partial x\partial y} - \frac{\partial^2 w}{\partial x^2}\Big(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w_0}{\partial y^2}\Big) - \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w_0}{\partial y^2}\Big], \tag{1.1}$$

$$D\Big(1 - \frac{\vartheta\zeta^2}{\beta}\nabla^2\Big)\nabla^2\nabla^2\chi - \frac{\partial^2 F}{\partial y^2}\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w_0}{\partial x^2} - k_{11}\Big) + 2\frac{\partial^2 F}{\partial x\partial y}\Big(\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 w_0}{\partial x\partial y}\Big) - \frac{\partial^2 F}{\partial x^2}\Big(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w_0}{\partial y^2} - k_{22}\Big) = q + (k_{11} + k_{22})\Psi. \tag{1.2}$$

The equations of the local stability of elastic thin sandwich shells are

* Numbers in the margin indicate pagination in the original foreign text.

$$\nabla^2\nabla^2 F = Eh\,\nabla_1^2(1 - h^2\beta^{-1}\nabla^2)\chi \tag{1.3}$$

$$D(1 - \vartheta h^2\beta^{-1}\nabla^2)\nabla^2\nabla^2\chi + \nabla_1^2 F -$$

$$-\left(N_{11}^0\frac{\partial^2}{\partial x^2} + 2N_{12}^0\frac{\partial^2}{\partial x\partial y} + N_{22}\frac{\partial^2}{\partial y^2}\right)(1 - h^2\beta^{-1}\nabla^2)\chi = 0 \tag{1.4}$$

$$w = (1 - h^2\beta^{-1}\nabla^2)\chi \tag{1.5}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{xy^2}, \qquad \nabla_1^2 = k_{11}\frac{\partial^2}{\partial y^2} + k_{22}\frac{\partial^2}{\partial x^2}.$$

Henceforth, expressions for the displacement components and specific moments will be needed to satisfy the boundary conditions.

The absolute shear of the boundary surfaces of the core is

$$a_1 = -\frac{h_3}{2}\frac{\partial}{\partial x}\left[1 + \frac{\theta_2 h^2}{\theta_1\beta}\Delta^2\right]\chi, \qquad a_2 = -\frac{h_3}{2}\frac{\partial}{\partial y}\left[1 + \frac{\theta_2 h^2}{\theta_1\beta}\nabla^2\right]\chi.$$

The total moment of normal forces of the load-carrying layers with respect to the center surface of the core as well as the moment of the core are /79

$$H_{11} = -D\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\vartheta_1\chi + {}^1/_2 ht_3(\gamma_1 - \gamma_2)\frac{\partial^2 F}{\partial y^2}$$

$$H_{22} = -D\left(\frac{\partial^2}{\partial y^2} + \nu\frac{\partial^2}{\partial x^2}\right)\vartheta_1\chi + {}^1/_2 ht_3(\gamma_1 - \gamma_2)\frac{\partial^2 F}{\partial x^2}$$

$$H_{12} = -D(1 - \nu)\frac{\partial^2}{\partial x\partial y}\vartheta_1\chi - {}^1/_2 ht_3(\gamma_1 - \gamma_2)\frac{\partial^2 F}{\partial x\partial y}.$$

The sum total of the intrinsic moments of the load-carrying layers is expressed as

$$M_{11} = -D\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\left(\vartheta_2 - \frac{\vartheta h^2}{\beta}\nabla^2\right)\chi + {}^1/_2 h(\gamma_1 t_1 - \gamma_2 t_2)\frac{\partial^2 F}{\partial y^2}$$

$$M_{22} = -D\left(\frac{\partial^2}{\partial y^2} + \nu\frac{\partial^2}{\partial x^2}\right)\left(\vartheta_2 - \frac{\vartheta h^2}{\beta}\nabla^2\right)\chi + {}^1/_2 h(\gamma_1 t_1 - \gamma_2 t_2)\frac{\partial^2 F}{\partial x^2}$$

$$M_{12} = -D(1 - \nu)\frac{\partial^2}{\partial x\partial y}\left(\vartheta_2 - \frac{\vartheta h^2}{\beta}\nabla^2\right)\chi - {}^1/_2 h(\gamma_1 t_1 - \gamma_2 t_2)\frac{\partial^2 F}{\partial x\partial y}.$$

The adjusted modulus of elasticity E, cylindrical rigidity D, and shear coefficient β will be determined by means of the rigidity characteristics and thicknesses of the load-carrying faces and the core. Let E_1, E_2, E_3 be the moduli of elasticity of the first kind of the load-carrying faces and the core, respectively; let ν_1, ν_2, ν_3 be the Poisson ratios of the materials of the load-carrying faces and the core; and let h_1, h_2, h_3 be the thicknesses of the load-carrying faces and the core. Let us then introduce the dimensionless rigidity characteristics and dimensionless thicknesses of the load-carrying faces and the core

$$\gamma_k = \frac{E_k h_k}{1 - \nu_k^2}\left(\sum_{i=1}^{3}\frac{E_i h_i}{1 - \nu_i^2}\right)^{-1}, \qquad t_k = \frac{h_k}{h}. \tag{1.6}$$

As implied by eq.(1.6),

$$\gamma_1 + \gamma_2 + \gamma_3 = 1, \qquad t_1 + t_2 + t_3 = 1 . \tag{1.7}$$

The adjusted Poisson ratio then becomes

$$\nu = \gamma_1\nu_1 + \gamma_2\nu_2 + \gamma_3\nu_3 . \tag{1.8}$$

The adjusted modulus of elasticity of the first kind will be determined from the formula

$$E = \frac{1-\nu^2}{h}\sum_{i=1}^{3}\frac{E_i h_i}{1-\nu_i^2} . \tag{1.9}$$

The cylindrical rigidity D and the shear coefficient β have the form of

$$D = \frac{Eh^3}{12(1-\nu^2)}\theta_0, \qquad \beta = \frac{12Gt_3(1-\nu^2)}{E\theta_1} . \tag{1.10}$$

Here, G is the shear modulus of the core material, while

$$\begin{aligned} \theta_0 &= \theta_1 + 2\theta_2 + \theta_3, \qquad \vartheta_1 = (\theta_1 + \theta_2)/\theta_0 \\ \vartheta &= (\theta_1\theta_3 - \theta_2^2)/\theta_0\theta_1, \qquad \vartheta_2 = (\theta_2 + \theta_3)/\theta_0 \\ \theta_1 &= t_3^2[1 + 2(\gamma_1 + \gamma_2) - 3(\gamma_1 - \gamma_2)^2] \\ \theta_2 &= 3\gamma_3 t_3(\gamma_1 t_1 + \gamma_2 t_2) + 6\gamma_1\gamma_2 t_3(t_1 + t_2) \\ \theta_3 &= 4(\gamma_1 t_1^2 + \gamma_2 t_2^2) - 3(\gamma_1 t_1 - \gamma_2 t_2)^2 . \end{aligned} \tag{1.11}$$

To determine the limits of variation in the coefficients, it is sufficient to consider the case of a shell of symmetric structure with a lightweight core.

For $\gamma_1 = \gamma_2 = 1/2$, $\gamma_3 = 0$, $t_1 = t_2 = t$ eq.(1.11) yields /80

$$\begin{gathered} \theta_1 = 3t_3^2, \quad \theta_2 = 3tt_3, \quad \theta_3 = 4t^2, \quad \theta_0 = 3(1-t)^2 + t^2 \\ \vartheta = [3(1 + t_3/t)^2 + 1]^{-1} . \end{gathered} \tag{1.12}$$

Hence, considering that $0 \le t \le 1/2$, $0 < t_3 < 1$, we have

$$1 \leqslant \theta_0 \leqslant 3, \qquad 0 < \vartheta \leqslant 1/4 . \tag{1.13}$$

As can be seen from this last estimate, ϑ is sufficiently small, e.g., for $t_3 = 0.6$ and $t = 0.2$ we have $\vartheta = 0.02$ which value decreases still further with increasing t_3 and γ_3. The coefficient ϑ/β characterizes the effect of the shear of the core on the intrinsic moments of the load-carrying faces; this effect may be disregarded for certain problems of stability and strength.

The boundary conditions for the functions F, χ ($x_1 = x_1^0$) will be as follows.

At the freely supported end ($N_{11} = \epsilon_2 = w = H_{11} = M_{11} = a_2 = 0$):

$$F = \nabla^2 F = \chi = \nabla^2\chi = \nabla^2\nabla^2\chi = 0 \, . \qquad (1.14)$$

At the clamped end ($N_{11} = \epsilon_2 = w = \partial w/\partial x = a_1 = H_{12} = 0$):

$$F = \nabla^2 F = \left(1 - \frac{h^2}{\beta}\nabla^2\right)\chi = \frac{\partial\chi}{\partial x} = \frac{\partial}{\partial x}\nabla^2\chi = 0 \, . \qquad (1.15)$$

At the end with clamped load-carrying faces, in the absence of connections that would prevent their relative shear ($N_{11} = \epsilon_2 = 0$, $w = \partial w/\partial x = H_{11} = 0$):

$$F = \nabla^2 F = \chi = \nabla^2\chi = \frac{\partial}{\partial x}\left(1 - \frac{h^2}{\beta}\nabla^2\right)\chi = 0 \, . \qquad (1.16)$$

At the end free of such connections ($N_{11} = 0$, $N_{12} = 0$, $H_{11} = M_{11} = 0$; Q sum = 0)

$$\begin{gathered} F = \frac{\partial F}{\partial x} = \left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\chi = \left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\nabla^2\chi = 0 \\ D\left[\frac{\partial^3}{\partial x^3} + (2-\nu)\frac{\partial^3}{\partial x\partial y^2}\right]\left(1 - \frac{\nu h^2}{\beta}\nabla^2\right)\chi - \\ -\left(N_{12}{}^\circ\frac{\partial^2}{\partial x\partial y} + N_{11}{}^\circ\frac{\partial^2}{\partial x^2}\right)\left(1 - \frac{h^2}{\beta}\nabla^2\right)\chi = 0 \, . \end{gathered} \qquad (1.17)$$

Note that conditions N_{11}°, N_{12}°, N_{22}° are considered as applied over the distance $e_0 = 1/2h(\beta_1 t_1 + \gamma_1 t_3 - \gamma_2 t_2 - \gamma_2 t_3)$ from the center surface of the core.

By introducing the solution function χ_1 from the formulas

$$\chi = \nabla^2\nabla^2\chi_1, \qquad F = Eh\nabla_1{}^2(1 - h^2\beta^{-1}\nabla^2)\chi_1 \qquad (1.18)$$

the system of equations (1.3), (1.4) can be reduced to a single stability equation

$$\begin{gathered} D(1 - \nu h^2\beta^{-1}\nabla^2)\nabla^2\nabla^2\nabla^2\nabla^2\chi_1 + Eh\nabla_1{}^2\nabla_1{}^2(1 - h^2\beta^{-1}\nabla^2)\chi_1 - \\ -\left(N_{11}{}^\circ\frac{\partial^2}{\partial x^2} + 2N_{12}{}^\circ\frac{\partial^2}{\partial x\partial y} + N_{22}{}^\circ\frac{\partial^2}{\partial y^2}\right)\left(1 - \frac{h^2}{\beta}\nabla^2\right)\nabla^2\nabla^2\chi_1 = 0 \, . \end{gathered} \qquad (1.19)$$

The above system of equations (1.3), (1.4), as well as eq.(1.19), were derived on the assumption that the angles of rotation of the normal and initial surface of the core may be expressed as partial derivatives of some function. The boundary conditions examined below correspond to this assumption.

2. Shell of Revolution /81

Consider a local stability loss in a shell of revolution supported on both sides. Assume that the shell is subject to an external normal pressure q, an axial compressive force N, and a torque M_t in the plane of a parallel circle;

under the action of these loads, the specific stresses

$$N_{11}^{\circ} = p_1, \qquad N_{22}^{\circ} = p_2, \qquad N_{12}^{\circ} = s \tag{2.1}$$

$$\begin{aligned} p_1 &= \frac{qR_2}{2}\left(1 - \frac{r_0^2}{R_2^2 \sin^2\alpha}\right) + \frac{N}{2\pi R_2 \sin^2\alpha} \\ p_2 &= \frac{qR_2}{2}\left(1 + \frac{r_0^2}{R_2^2 \sin^2\alpha}\right) - \frac{N}{2\pi R_2 \sin^2\alpha}, \qquad s = \frac{M_{torque}}{2\pi R_2^2 \sin^2\alpha} \end{aligned} \tag{2.2}$$

will arise in the subcritical momentless state.

Here, r_0 is the radius of the parallel circle and α is the angle between the axis and the normal to the initial surface.

Then the problem reduces to the solution of the stability equation

$$\begin{aligned} &D(1 - \vartheta h^2\beta^{-1}\nabla^2)\nabla^2\nabla^2\nabla^2\nabla^2\chi_1 + Eh\nabla_1^2\nabla_1^2(1 - h^2\beta^{-1}\nabla^2)\chi_1 + \\ &+ \left(p_1\frac{\partial^2}{\partial x^2} - 2s\frac{\partial^2}{\partial x \partial y} + p_2\frac{\partial^2}{\partial y^2}\right)\left(1 - \frac{h^2}{\beta}\nabla^2\right)\nabla^2\nabla^2\chi_1 = 0 \end{aligned} \tag{2.3}$$

with the boundary conditions (for x = 0 and x = ℓ) of free support

$$\chi_1 = \nabla^2\chi_1 = \nabla^2\nabla^2\chi_1 = \nabla^2\nabla^2\nabla^2\chi_1 = \nabla^2\nabla^2\nabla^2\nabla^2\chi_1 = 0 \ . \tag{2.4}$$

We seek the solution of eq.(2.3) in the form

$$\chi_1 = \sin(m\pi x / l)\{a \sin[n(y - \eta x) / R_2] + b \cos[n(y - \eta x) / R_2]\}. \tag{2.5}$$

Here, m is the number of half-waves along the generatrix of the shell; n is the number of waves along the perimeter; η is the parameter characterizing the inclination of helical loops due to the action of the torque; and a, b are constants.

Introducing eq.(2.5) into eq.(2.3), we obtain

$$\begin{aligned} &a[\psi(n, -m) - \psi(n, m)]\cos(m\pi x / l)\cos[n(y - \eta x) / R_2] + \\ &+ a[\psi(n, -m) + \psi(n, m)]\sin(m\pi x / l)\sin[n(y - \eta x) / R_2] + \\ &+ b[\psi(n, m) - \psi(n, -m)]\sin(m\pi x / l)\cos[n(y - \eta x) / R_2] + \\ &+ b[\psi(n, m) + \psi(n, -m)]\cos(m\pi x / l)\sin[n(y - \eta x) / R_2] = 0, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \psi(n, m) = {} & D\frac{1 + \vartheta h^2\beta^{-1}[(-n\eta/R_2 + m\pi/l)^2 + (n/R_2)^2]}{1 + h^2\beta^{-1}[(-n\eta/R_2 + m\pi/l)^2 + (n/R_2)^2]}\left[\left(-\frac{n\eta}{R_2} + \frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2 + \\ &+ \frac{Eh}{R_2^2}\frac{[(-n\eta/R_2 + m\pi/l)^2 + (R_2/R_1)(n/R_2)^2]^2}{[(-n\eta/R_2 + m\pi/l)^2 + (n/R_2)^2]^2} - \\ &- p_1\left(-\frac{n\eta}{R_2} + \frac{m\pi}{l}\right)^2 - 2s\frac{n}{R_2}\left(-\frac{n\eta}{R_2} + \frac{m\pi}{l}\right) - p_2\frac{n^2}{R_2^2} = 0 \ . \end{aligned} \tag{2.7}$$

The equality (2.6) is possible only on condition that

$$\psi(n, m) = 0, \qquad \psi(n, -m) = 0 \ . \tag{2.8}$$

Thus, we have two equations for finding the critical combination of stresses p_1, p_2, s. We will now consider particular problems.

3. Cylindrical Shell /82

a) Uniform Axial Compression p_1

Assuming that $R_1 \to \infty$, $R_2 = R$, $T_1 = p_2 = s = 0$, we obtain from eq.(2.7) the formula for the critical compressive axial unit stress

$$p_1 = D\,\frac{1 + \vartheta h^2\beta^{-1}m_1}{1 + h^2\beta^{-1}m_1}\,m_2 + \frac{Eh}{R^2 m_2} \tag{3.1}$$

$$m_1 = \frac{m^2\pi^2}{l^2} + \frac{n^2}{R^2}, \qquad m_2 = \frac{l^2}{m^2\pi^2}\left(\frac{m^2\pi^2}{l^2} + \frac{n^2}{R^2}\right)^2 , \tag{3.2}$$

with $(m\pi/\ell)^2 \le m_1 < \infty$, $m_1 \le m_2 < \infty$. According to eq.(3.1), p_1 reaches its minimum at the boundary of variation in the variables m_1, m_2, determined by the equation $m_1 = m_2$, since the function $(1 + vh^2\beta^{-1}m_1)/(1 + h^2\beta^{-1}m_1)$ is monotonic. We seek the minimum of p_1 on differentiating the expression

$$p_1 = D\,\frac{1 + \vartheta h^2\beta^{-1}m_1}{1 + h^2\beta^{-1}m_1}\,m_1 + \frac{Eh}{R^2 m_1} \tag{3.3}$$

with respect to m_1. This leads to the equation

$$Dm_1^2\left(1 + \frac{2\vartheta h^2}{\beta}m_1 + \frac{\vartheta h^4}{\beta^2}m_1^2\right) - \frac{Eh}{R^2}\left(1 + \frac{h^2}{\beta}m_1\right)^2 = 0 \ . \tag{3.4}$$

This equation contains one sign reversal and thus has one positive root realizing the minimum of eq.(3.3). Considering that ϑ is a negligible quantity, we determine m_1 by solving the quadratic equation

$$\left(1 - \frac{Eh^5}{DR^2\beta^2}\right)m_1^2 - \frac{2Eh^3}{R^2D\beta}\,m_1 - \frac{Eh}{DR^2} = 0 \tag{3.5}$$

whose positive root will be

$$m_1 = \frac{2\sqrt{3(1-\nu^2)}}{Rh\sqrt{\theta_0}}\left(1 - \frac{2h\sqrt{3(1-\nu^2)}}{R\beta\sqrt{\theta_0}}\right) , \tag{3.6}$$

while its corresponding least critical unit stress will be

$$p_1 = \frac{Eh^2\sqrt{\theta_0}}{R\sqrt{3(1-\nu^2)}} - \frac{E^2h^3\theta_1}{12Gt_3R^2(1-\nu^2)} \ . \tag{3.7}$$

The limit of applicability of eqs.(3.6), (3.7) is determined by the inequality

$$1-\frac{2h\sqrt{3(1-\nu^2)}}{R\beta\sqrt{\theta_0}}>\sqrt{\vartheta}\ . \tag{3.8}$$

Hence, beyond the limits of this inequality, given the minimization of expression (3.3), allowance must be made for ϑ; however, in accordance with eq.(3.6) the parameter m_1 is extremely large in this case and $h/R\beta \gtrsim 1$. Hence, eq.(3.3) may be rewritten as

$$p_1 = D\beta h^{-2}(1+\nu h^2\beta^{-1}m_1)+Eh(R^2m_1)^{-1}\ . \tag{3.9}$$

The condition $dp_1/dm_1 = 0$ yields

$$m_1=\frac{2\sqrt{3(1-\nu^2)}}{Rh\sqrt{\theta_0\vartheta}}\ , \tag{3.10}$$

so that the critical unit stress will become

$$p_1=\frac{Eh^2\sqrt{\theta_0\vartheta}}{R\sqrt{3(1-\nu^2)}}+Gh\frac{\theta_0 t_3}{\theta_1}\ . \tag{3.11}$$

b) <u>External Transverse Pressure q</u>

In eq.(2.7), we put

$$R_1=\infty,\quad R_2=R,\quad \eta=p_1=s=0,\quad p_2=qR,\quad m=1$$
$$\lambda=l/R,\quad k=h^2\pi^2/\beta R^2,\quad \mu^2=12(1-\nu^2)R^2/\pi^4h^2\theta_0\ .$$

We have (Figs.1, 2; these and the following diagrams give the minimal values of the critical load) /83

$$p_2=\frac{12(1-\nu^2)qR^3}{Eh^3\theta_0\pi^2}=\frac{\lambda^2\pi^2+k\vartheta(\pi^2+\lambda^2\pi^2)}{\lambda^2\pi^2+k(\pi^2+\lambda^2\pi^2)}\frac{\pi^2}{n^2}\left(\frac{1}{\lambda^2}+\frac{n^2}{\pi^2}\right)^2+\frac{\mu^2\pi^6}{n^2(\pi^2+\lambda^2n^2)^2}\ . \tag{3.12}$$

c) <u>External Omnilateral Pressure q</u>

In eq.(2.7), we put

$$R_1=\infty,\ R_2=R,\ p_1={}^1/_2qR,\ p_2=qR,\ s=0,\ m=1,\ \lambda=l/R,$$
$$k=h^2\pi^2/\beta R^2,\qquad \mu^2=12(1-\nu^2)R^2/\pi^2h^2\theta_0.$$

We have (Figs.3, 4)

$$q^*=\frac{12(1-\nu^2)qR^3}{Eh^3\theta_0\pi^2}=\frac{1+k\vartheta(1/\lambda^2+n^2/\pi^2)}{1+k(1/\lambda^2+n^2/\pi^2)}\frac{(1/\lambda^2+n^2/\pi^2)^2}{{}^1/_2\lambda^2+n^2/\pi^2}+$$
$$+\frac{\mu^2}{\lambda^4}\frac{1}{({}^1/_2\lambda^2+n^2/\pi^2)(1/\lambda^2+n^2/\pi^2)^2}\ . \tag{3.13}$$

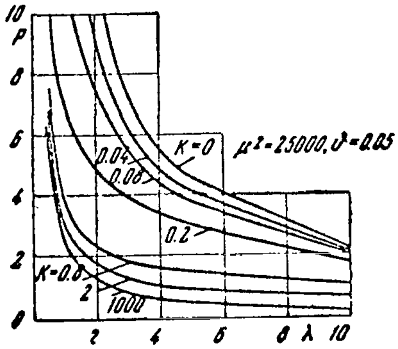

Fig.1

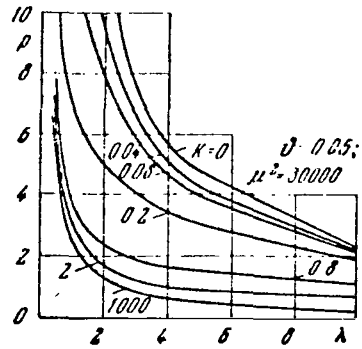

Fig.2

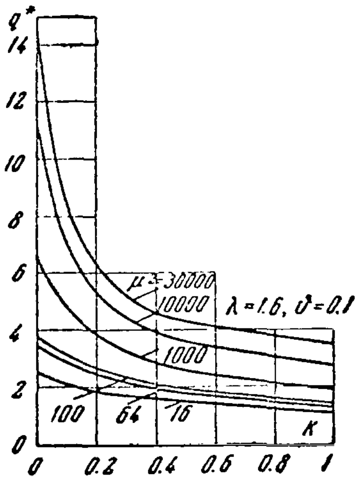

Fig.3

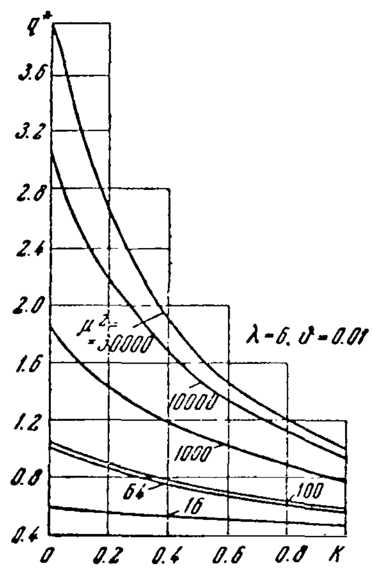

Fig.4

d) <u>Torsion</u> /84

From eq.(2.7), at

$$R_1 = \infty,\ R_2 = R,\ p_1 = p_2 = 0,\ k = h^2\pi^2 / l^2\beta,\ \varepsilon = Rh / l^2,$$
$$\lambda = l / R,\ \eta_1 = \eta\varepsilon^{-1/4},\ \xi = (\pi R / ln)\,\varepsilon^{-1/4},\ t = h / R,\ \theta = \theta_0 / (1 - \nu^2)$$

we derive two equations for determining s

$$s = \frac{Eh^2}{2R}\varepsilon^{1/4}\left\{\frac{\pi^2\theta}{12\xi^2}\,\frac{1 + \theta k\varepsilon^{-1/2}\xi^{-2}[\varepsilon^{1/2}(-\eta_1 \pm \xi)^2 + 1]}{1 + k\varepsilon^{-1/2}\xi^{-2}[\varepsilon^{1/2}(-\eta_1 \pm \xi)^2 + 1]}\times\right.$$
$$\left.\times\frac{[\varepsilon^{1/2}(-\eta_1 \pm \xi)^2 + 1]^2}{-\eta_1 \pm \xi} + \frac{\xi^2}{\pi^2}\,\frac{(-\eta_1 \pm \xi)^2}{[\varepsilon^{1/2}(-\eta_1 \pm \xi)^2 + 1]^2}\right\}. \qquad (3.14)$$

For shells with a small parameter ϵ, eq.(3.14) is simplified, since then it may be assumed that

$$\varepsilon^{1/2}(-\eta_1 \pm \xi)^2 + 1 \sim 1 \; . \tag{3.15}$$

Then, we find

$$s = \frac{Eh^2}{2R}\varepsilon^{1/4}\left[\frac{\pi^2\theta}{12\xi^2(-\eta_1+\xi)}\,\frac{1+\vartheta k(\xi^2\varepsilon^{1/2})^{-1}}{1+k(\xi^2\varepsilon^{1/2})^{-1}} + \frac{\xi^2}{\pi^2}(-\eta_1+\xi)^3\right]$$
$$s = \frac{Eh^2}{2R}\varepsilon^{1/4}\left[\frac{\pi^2\theta}{12\xi^2(-\eta_1-\xi)}\,\frac{1+\vartheta k(\xi^2\varepsilon^{1/2})^{-1}}{1+k(\xi^2\varepsilon^{1/2})^{-1}} + \frac{\xi^2}{\pi^2}(-\eta_1-\xi)^3\right].$$

Adding and subtracting these equations, we have

$$s = \frac{2Eh^2}{R\pi^2}\varepsilon^{1/4}, \qquad s = \frac{2Eh^2}{R\pi^2}\varepsilon^{1/4}\eta_1\xi^2(\eta_1^2+\xi^2) \tag{3.16}$$

$$\eta_1^4 - \frac{2}{3}\xi^2\eta_1^2 - \frac{\xi^4}{3} - \frac{\pi^4\theta}{36\xi^4}\,\frac{1+\vartheta k(\xi^2\varepsilon^{1/2})^{-1}}{1+k(\xi^2\varepsilon^{1/2})^{-1}} = 0 \; . \tag{3.17}$$

Equation (3.17) makes it possible to find η_1 from a given n (Figs.5 - 6).

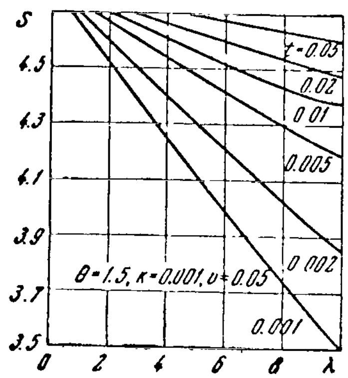

Fig.5

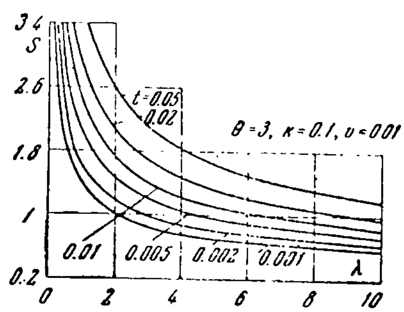

Fig.6

e) <u>Eccentric Compression</u>

The pressure over the perimeter of the shell face, subjected to the action of the eccentric axial force N, is found from the formula

$$p_1 = p_0\left(1 + 2\varepsilon\cos\frac{ny}{R}\right), \qquad p_0 = \frac{N}{2\pi R}, \qquad \varepsilon = \frac{e}{R}, \tag{3.18}$$

where e is the eccentricity of the force.

Introducing

$$\chi_1 = \sin\frac{m\pi x}{l}\sum_{-\infty}^{+\infty} a_n e^{iny/R} \tag{3.19}$$

into eq.(2.3), we have /85

$$a_{n+1}\varepsilon - (\psi_n\lambda - 1)a_n + a_{n-1}\varepsilon = 0 \qquad (\lambda = p_0^{-1}) . \tag{3.20}$$

Here,

$$\psi_n = D\,\frac{1+\nu h^2\beta^{-1}(m^2n^2/l^2 + n^2/R^2)}{1+h^2\beta^{-1}(m^2\pi^2/l^2 + n^2/R^4)}\,\frac{(m^2n^2/l^2 + n^2/R^2)^2}{m^2\pi^2/l^2} +$$
$$+\frac{Eh}{R^2}\,\frac{m^3\pi^2/l^3}{(m^2\pi^2/l^2 + n^2/\pi^2)^2} . \tag{3.21}$$

The equality (3.20) can be written as

$$\frac{a_n}{a_{n-1}} = \frac{\varepsilon}{\lambda\psi_n - 1 - \varepsilon a_{n+1}/a_n} \quad \text{for } n \geqslant 0$$
$$\frac{a_n}{a_{n+1}} = \frac{\varepsilon}{\lambda\psi_n - 1 - \varepsilon a_{n-1}/a_n} \quad \text{for } n < 0 . \tag{3.22}$$

Since $\psi_n = O(n^4)$ when $n \to \pm\infty$, the series (3.19) converges. Considering that $\psi_n = \psi_{-n}$, we have

$$\frac{a_n}{a_{n-1}} = \frac{a_{-n}}{a_{-n+1}}, \qquad \frac{a_0}{a_{-1}} = \frac{a_0}{a_1} .$$

Therefore,

$$\lambda\psi_0 = 1 + \varepsilon\left(\frac{a_1}{a_0} + \frac{a_{-1}}{a_0}\right) = 1 + 2\varepsilon\frac{a_1}{a_0} . \tag{3.23}$$

However, $a_1/a_0 \leq 1$ so that

$$p_0 \geqslant \psi_0/(1+2\varepsilon) . \tag{3.24}$$

For sufficiently large n

$$\frac{a_n}{a_{n-1}} \approx \frac{\varepsilon}{\lambda\psi_n - 1} . \tag{3.25}$$

Returning to a_0, we have an infinite continued fraction

$$\frac{a_1}{a_0} = \frac{\varepsilon}{(\psi_1\lambda - 1) - \frac{\varepsilon^2}{(\psi_2\lambda - 1) - \dots}} \quad . \tag{3.26}$$

Now, on the basis of eq.(3.22), we obtain the infinite continued fraction

$$\frac{\lambda\psi_0 - 1}{2} = \left\{ \frac{\varepsilon^2}{(\lambda\psi_1 - 1) -}, \frac{\varepsilon^2}{(\lambda\psi_2 - 1) -}, \frac{\varepsilon^2}{(\lambda\psi_3 - 1) -} \dots \right\} , \tag{3.27}$$

which makes it possible to determine both λ for a given ϵ and ϵ for a given λ. Numerical calculations show that when determining ϵ for a given λ in fractions of ψ_0, p_0 it is sufficient to confine the consideration to three or four terms of the fraction (3.27). Consideration of the first four terms leads to a quadratic equation with respect to ϵ^2 and hence presents no computational difficulties. Calculations performed over a sufficiently broad range of parameters show that p_0 can be determined with high accuracy and always with a margin, by means of the formula

$$p_0 = \psi_0 / (1 + 2\varepsilon) \ . \tag{3.28}$$

The same finding was originally obtained by Fluegge in examining an analogous problem for homogeneous eccentrically compressed cylindrical shells.

f) Effect of Compound Stresses /86

The critical stresses in the presence of compound axial stress, bending stress, and transverse pressure satisfy the condition

$$\frac{q_*^*}{q_{0*}^*} + \frac{p_*}{p_{0*}} + \left(\frac{S_*}{S_{0*}}\right)^2 = 1 \ , \tag{3.29}$$

where q_{0*}^*, p_{0*}, S_{0*} are, respectively, the parameters of the critical uniform transverse (or omnilateral) pressure, uniform axial pressure, and tangential boundary stress, with each stress acting separately; q_*^*, p_*, S_* are the respective critical values of these parameters when the stresses are combined.

g) Oscillations and Stability of a Shell Clamped on Both Sides

On approximating buckling by the function

$$w = W\left[\cos\frac{(m-1)\pi x}{l} - \cos\frac{(m+1)\pi x}{l}\right]\cos\frac{ny}{R}\cos\omega t \ , \tag{3.30}$$

we obtain the expression for χ_1 in the form of

$$\chi_1 = \frac{WR^4}{\pi^4}\left\{\frac{\cos[(m-1)\pi x/l]}{[1+k((m-1)^2/\lambda^2+n^2/\pi^2)][(m-1)^2/\lambda^2+n^2/\pi^2]^2} - \right.$$
$$\left. - \frac{\cos[(m+1)\pi x/l]}{[1+k((m+1)^2/\lambda^2+n^2/\pi^2)][(m+1)^2/\lambda^2+n^2/\pi^2]^2}\right\}\cos\frac{ny}{R}\cos\omega t \, . \tag{3.31}$$

Substituting this function in eq.(2.3), complemented on its left-hand side by the inertia term

$$\left(\sum_{k=1}^{3}\rho_k h_k\right)\frac{\partial^2}{\partial t^2}\nabla^2\nabla^2(1-h^2\beta^{-1}\nabla^2)\chi_1 \quad , \tag{3.32}$$

(where ρ_k is the specific density of the material of the faces), and performing orthogonalization with respect to w, we have

$$p_1^* \left[(1+\delta_{1m})\frac{(m-1)^2}{\lambda^2} + \frac{(m+1)^2}{\lambda^2}\right] + p_2^*(2+\delta_{1m})\frac{n^2}{\pi^2} + \omega_*^2(2+\delta_{1m}) =$$
$$= \frac{1+\vartheta k[(m-1)^2/\lambda^2+n^2/\pi^2]}{1+k[(m-1)^2/\lambda^2+n^2/\pi^2]}\left[\frac{(m-1)^2}{\lambda^2}+\frac{n^2}{\pi^2}\right]^2(1+\delta_{1m}) +$$
$$+ \frac{1+\vartheta k[(m+1)^2/\lambda^2+n^2/\pi^2]}{1+k[(m+1)^2/\lambda^2+n^2/\pi^2]}\left[\frac{(m+1)^2}{\lambda^2}+\frac{n^2}{\pi^2}\right] +$$
$$+ \frac{\mu^2}{\lambda^4}\frac{(m-1)^4(1+\delta_{1m})}{[(m-1)^2/\lambda^2+n^2/\pi^2]^2} + \frac{\mu^2}{\lambda^4}\frac{(m+1)^4}{[(m+1)^2/\lambda^2+n^2/\pi^2]^2} \quad , \tag{3.33}$$

where

$$p_1^* = p_1\frac{R^2}{D\pi^2}, \qquad p_2^* = p_2\frac{R^2}{D\pi^2}, \qquad \omega_*^2 = \frac{\Omega\omega^2\pi^4}{DR^4}, \qquad \Omega = \sum_{k=1}^{3}\rho_k h_k ,$$
$$k = \frac{h^2\pi^2}{\beta R^2}, \qquad \lambda = \frac{l}{R}, \qquad \mu^2 = \frac{12(1-\nu^2)R^2}{\pi^4 h^2\theta_0}, \qquad \delta_{1m} = \begin{cases}0 & m \neq 1\\ 1 & m = 1\end{cases} .$$

For the case of a cylindrical shell with a structure symmetric over its thickness, see elsewhere (Bibl.2).

4. Spherical Shell Subject to External Normal Pressure q

Assuming in eq.(2.7) that $R_1 = R_2 = R$, $s = 0$, $p_1 = p_2 = 1/2\ qR$, we have

$$\frac{qR}{2} = D\frac{\beta+\vartheta h^2\lambda}{\beta+h^2\lambda}\lambda + \frac{Eh}{R^2\lambda} \; . \tag{4.1}$$

Investigation of the minimum on the basis of eq.(4.1) was performed by studying the stability of a cylindrical shell for the case of axial compression. We have

$$q = \frac{2Eh^2\sqrt{\theta_0}}{R^2\sqrt{3(1-\nu^2)}} - \frac{2E^2h^3\theta_1}{12Gt_zR^2(1-\nu^2)} \quad \text{at} \quad 1 - \frac{2h\sqrt{3(1-\nu^2)}}{R\beta\sqrt{\theta_0}} < \sqrt{\theta} \; . \tag{4.2}$$

Here, λ is determined by the formula /87

$$\lambda = \frac{2\sqrt{3(1-\nu^2)}}{Rh\sqrt{\theta_0}}\,\frac{1}{1-(2h\sqrt{3(1-\nu^2)})(R\beta\sqrt{\theta_0})^{-1}} \quad , \tag{4.3}$$

while, in the other cases,

$$q = \frac{2Eh^2\sqrt{\theta_0\vartheta}}{R^2\sqrt{3(1-\nu^2)}} + 2G\frac{h}{R}\frac{\theta_0 t_3}{\theta_1}, \qquad \lambda = \frac{1}{\sqrt{\vartheta\theta_0}}\frac{2\sqrt{3(1-\nu^2)}}{Rh}. \tag{4.4}$$

For the case of a spherical shell with a structure symmetric over its thickness and with a lightweight core, see elsewhere (Bibl.3).

5. Circular Conical Shell Exposed to Uniform Axial Compression and Uniform External Normal Pressure

Let r_0, r_1 be the distances along the cone generatrix to the upper and lower bases, respectively; let α be the half-angle of aperture of the cone;

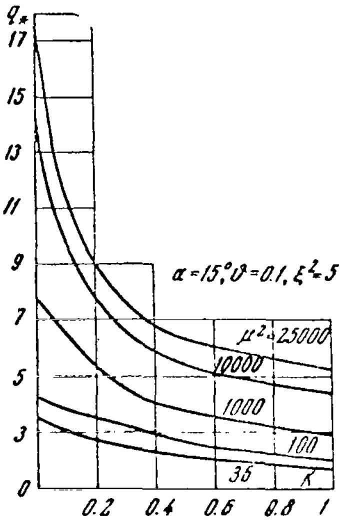

Fig.7

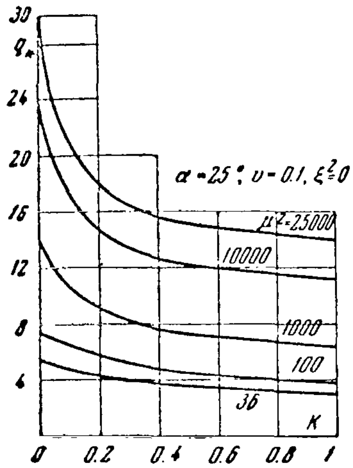

Fig.8

$\ell = r_1 - r_0$ is the length of the shell along the generatrix; r is the coordinate measured from the vertex of the cone; φ is the polar angle; q is the external normal pressure; N is the axial stress applied to the lower base; R is the radius of the lower base. The other notations are the same as above.

Let $r = r_1 \exp(\pi\zeta x)$, where (Bibl.4) $\zeta = \pi^{-1}\ln(r_0/r_1)$; therefore, $0 \leq x \leq 1$ and $-\infty \leq \zeta \leq 0$.

$$w = w_1(x)\cos n\varphi, \qquad \chi = \chi_1(x)\cos n\varphi, \qquad F = F_1(x)\cos n\varphi\ . \tag{5.1}$$

Using the Bubnov-Galerkin method, we obtain

$$\int_0^1 \left[\nabla_n^2\nabla_n^2 F_1 - Eh\cot\alpha\,(\pi\zeta r_1)^{-3} e^{-3\zeta\pi x}\left(\frac{d^2}{dx^2} - \pi\zeta\frac{d}{dx}\right)w_1\right]\delta F_1 e^{2\zeta\pi x}\,dx = 0 \tag{5.2}$$

$$\int_0^1 \left[\left(1-\frac{h^2}{\beta}\nabla_n^2\right)\chi_1 - w_1\right]\delta\chi_1 e^{2\pi\zeta x}\,dx = 0 \tag{5.3}$$

$$\begin{aligned}\int_0^1 \Big\{ & D\nabla_n^2\nabla_n^2\left(1-\frac{\vartheta h^2}{\beta}\nabla_n^2\right)\chi_1 + \cot\alpha\,(\pi\zeta r_1)^{-3}e^{-3\zeta\pi x}\left(\frac{d^2}{dx^2}-\pi\zeta\frac{d}{dx}\right)F_1 + \\ & + {}^1\!/_2\, q r_1 \tan\alpha\,(\zeta\pi r_1)^{-2}e^{-\pi\zeta x}\left(\frac{d^2}{dx^2}+\pi\zeta\frac{d}{dx}-\frac{2n^2\pi^2\zeta^2}{\sin^2\alpha}\right)w_1 + \\ & + N(\pi\zeta r_1)^{-2}e^{-3\pi\zeta x}\left(\frac{d^2}{dx^2}-\pi\zeta\frac{d}{dx}\right)w_1\Big\}\delta w_1 e^{2\pi\zeta x}\,dx = 0\ ,\end{aligned} \tag{5.4}$$

where /88

$$\nabla_n^2 = (\zeta r_1\pi)^{-2}e^{-2\pi\zeta x}\left(\frac{d^2}{dx^2}-\frac{n^2\pi^2\zeta^2}{\sin^2\alpha}\right).$$

For a shell freely supported on both sides, the condition

$$w_1 = [1-(h^2/\beta)\nabla_n^2]\chi_1 = 0$$

will be the only kinematic boundary condition (at x = 0, x = 1).

In investigations on local stability loss, neglecting the static boundary conditions for moderately short shells introduces no great error; hence we prescribe the functions w_1, χ_1, F_1 in the form (m being the number of half-waves along the cone element)

$$w_1 = w_0 e^{3\zeta\pi x}\sin m\pi x, \qquad \chi_1 = \chi_0 e^{3\zeta\pi x}\sin m\pi x, \qquad F_1 = F_0 e^{4\zeta\pi x}\sin m\pi x\ . \tag{5.5}$$

Using eqs.(5.2) - (5.4), we have

$$\begin{aligned}&\frac{qR^2}{D\pi^2\cos\alpha}\lambda^2\frac{2}{7}\frac{1-e^{7\pi\zeta}}{1-e^{4\pi\zeta}}\frac{2M-12.5\zeta^2-m^2}{12.25\zeta^2+m^2}+\frac{NR^2}{D\pi^2}\lambda^2\frac{4}{5}\frac{1-e^{6\pi\zeta}}{1-e^{4\pi\zeta}}\frac{6.5\zeta^2+m^2}{6.25\zeta^2+m^2}= \\ &=\left\{\left[1+k\lambda^{-2}\frac{4}{3}\frac{1-e^{6\pi\zeta}}{1-e^{8\pi\zeta}}\frac{(m^2+16\zeta^2)(M+9\zeta^2)}{m^2+9\zeta^2}\right]^{-1}\times\right. \\ &\times\left[\frac{(M-\zeta^2)(M-9\zeta^2)+16\zeta^2(M-3\zeta^2)-12m^2\zeta^2}{m^2+4\zeta^2}+\right. \\ &\left.\left.+2\vartheta k\lambda^{-2}\frac{1-e^{2\pi\zeta}}{1-e^{4\pi\zeta}}\frac{(M-3\zeta^2)(M-\zeta^2)^2+4\zeta^2m^2(M-3\zeta^2)}{m^2+\zeta^2}\right]\right\}+ \\ &+\mu^2\lambda^4\cos^2\alpha\frac{2(1-e^{6\pi\zeta})}{3(1-e^{4\pi\zeta})}\frac{m^2+9\zeta^2}{[(M-16\zeta^2)(M-4\zeta^2)+36\zeta^2(M-8\zeta^2)-32m^2\zeta^2]}\ ,\end{aligned} \tag{5.6}$$

where

$$\mu^2=\frac{12R^2(1-\nu^2)}{h^2\vartheta_0\pi^4},\qquad \lambda^2=\frac{\zeta^2\pi^2}{\sin^2\alpha},\qquad k=\frac{h^2\pi^2}{\beta R^2},\qquad M=m^2+\frac{n^2\lambda^2}{\pi^2}.$$

Change-over to a cylindrical shell begins at $\alpha \to 0$; then, $\zeta \to 0$, $\lambda \to -\ell/R$. For a cone closed at the vertex $\xi^2 = \zeta^2 \to 0$, eq.(5.6) assumes the form (Figs.7, 8)

$$q_* = \frac{qR^3}{D\pi^2} = \frac{5.36\cos\alpha}{(n_1^2 - 6.25)\lambda_1^2}\left\{\left[1 + \frac{64}{27}\frac{k}{\lambda_1^2}(n_1^2 + 9)\right]^{-1} \times \right.$$

$$\left. \times \left[(n_1^4 + 6n_1^2 - 39) + \frac{80k}{\lambda_1^2}(n_1^2 - 3)(n_1^2 - 1)^2\right] + \frac{24\mu^2\lambda_1^4\cos^2\alpha}{n_1^4 + 16n_1^2 - 224}\right\}, \tag{5.7}$$

where

$$\lambda_1^2 = \frac{\pi^2}{\sin^2\alpha}, \qquad n_1^2 = \frac{n^2}{\sin^2\alpha}.$$

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